

COORDINATE CALCULI ON ASSOCIATIVE ALGEBRAS*

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Abstract

A new notion of an optimal algebra for a first order free differential was introduced in [1]. Some relevant examples are indicated. Quadratic identities in the optimal algebras and calculi on quadratic algebras are studied. Canonical construction of a quantum de Rham complex for the coordinate differential is proposed. The relations between calculi and various generalizations of the Yang–Baxter equation are established.

1 Introduction.

Quantum spaces are identified with noncommutative algebras. This idea generalizes the concept of supersymmetry and opens new possibilities for quantization of classical systems. New objects like quantum groups and quantum spaces are usually considered as (multi)parametric noncommutative (=quantum) deformations of the corresponding classical (=commutative) objects.

Differential calculi on quantum spaces have been elaborated by Pusz and Woronowicz [10, 9] and by Wess and Zumino [12]. Bicovariant differential calculi on quantum groups were presented by Woronowicz [13]. Woronowicz found that construction of first order calculus, a bimodule of one forms is not functorial. There are many nonequivalent calculi for a given associative algebra. This yields a problem of classification of first order calculi.

In our previous paper [1] following the developments by Pusz, Woronowicz [10] and by Wess, Zumino [12], we have proposed a general algebraic formalism for a first

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order calculus on an arbitrary associative algebra with a given generating space. A basic idea was that a noncommutative differential calculus is best handled by means of commutation relations between generators of an algebra and its differentials. We assume that a bimodule of one forms is a free right module. This allows to define partial derivatives (vector fields). Corresponding calculi and a differentials are said to be *coordinate calculi* and *coordinate differentials*.

Throughout this paper, \mathbb{F} will denote a field. An algebra means a linear associative unital algebra over the field \mathbb{F} . A coordinate algebra is an algebra R together with a chosen set of generators x^1, \dots, x^n (coordinates). We assume that the coordinates are linearly independent but in general they do not need to be algebraically independent. In other words, coordinate algebra is an algebra together with its presentation. A presentation of an algebra R is an epimorphism $\pi : \mathbb{F} \langle x^1, \dots, x^n \rangle \rightarrow R$, where $\mathbb{F} \langle x^1, \dots, x^n \rangle$ is a free algebra generated by the variables x^1, \dots, x^n . Let $I_R = \ker \pi$, then $R \cong \mathbb{F} \langle x^1, \dots, x^n \rangle / I_R$ where, I_R is an (two-sided) ideal of relations in R and $\pi(x^i)$ are generators of R . By abuse of notation we continue to write x^i for $\pi(x^i)$ when no confusion can arise. The free algebra is isomorphic to a tensor algebra TV of a vector space $V = \text{lin}(x^1, \dots, x^n)$. If V is infinite dimensional then we have infinite number of generators (x^1, x^2, \dots) .

Coordinate algebra is interpreted as an algebra of (polynomial) functions $R = \text{Func}(X)$ on quantum space X . This resembles the situation from algebraic geometry. Homogeneous ideals corresponds to graded algebras (projective case). Roughly speaking coordinate algebras are quantum versions of algebraic varieties.

2 Coordinate calculi and optimal algebras.

In this section we mainly review our results from [1]. For the proofs we refer the reader to [1].

Differential or (first order differential) calculus is a linear mapping from an algebra R to a R -bimodule M satisfying the Leibniz rule:

$$d(uv) = d(u)v + ud(v) \quad (1)$$

Definition 2.1 Let R be an algebra with a generating space $V = \text{lin}(x^1, \dots, x^n)$. A differential $d : R \rightarrow M$ and the corresponding calculus is said to be *coordinate* if the bimodule M is a free right R -module freely generated by dx^1, \dots, dx^n .

This definition essentially depends on the generating space V . The bimodule M as a right module is isomorphic to the right module $V \otimes R$. Let $v \in R$. The left multiplication $u \mapsto vu$ is an endomorphism of the right module M . Ring of all endomorphisms of any free module of rank n is isomorphic to the ring $R_{n \times n}$ of all n by n matrices with entries from R . Therefore, we can find an algebra homomorphism $A : R \rightarrow R_{n \times n}$ defined by the formulae

$$v dx^i = dx^k \cdot A(v)_k^i. \quad (2)$$

Clearly, it satisfies the homomorphism property

$$A(uv)_k^i = A(u)_k^l A(v)_l^i. \quad (3)$$

Example 2.2 (*Universal differentials*) Let us consider the case V is any supplement of the subspace $\mathbb{F} \cdot 1$, i.e. $R = V \oplus \mathbb{F} \cdot 1$. Of course if R is not a finite dimensional then we have an infinite number of generators. Nevertheless there exists a coordinate differential with respect to this space of generators. It is exactly the universal differential (cf. e.g. [7, 13]). Let $M(R) \subset R \otimes R$ denotes a kernel of the multiplication map. Clearly, $M(R)$ is a bisubmodule. For any $u \in R$ we put $du = 1 \otimes u - u \otimes 1$. Decomposing $u = u_0 \cdot 1 + \sum_{i \geq 1} x^i u_i$ with $u_i \in \mathbb{F}$ we can find that $du = \sum_{i \geq 1} dx^i u_i$ is a unique decomposition. This means that $d : R \rightarrow M(R)$ is a coordinate calculus. The homomorphism A of the universal differential has the form

$$A(x^i)_k^j = C_k^{ij} - \delta_k^i x^j$$

where, the scalar coefficients C_k^{ij} come from the multiplication table for the generators

$$x^i x^j = C_k^{ij} x^k + D^{ij} \cdot 1. \quad (4)$$

Then the homomorphism property (3) is equivalent to associativity constraints of the structure constants C_k^{ij}, D^{ij} vis.

$$C_r^{ij} C_m^{rk} + D^{ij} \delta_m^k = C_r^{jk} C_m^{ir} + D^{jk} \delta_m^i.$$

If d is a coordinate differential, then linear maps $D_k : R \rightarrow R$ (partial derivatives with respect to the coordinates (x^1, \dots, x^n)) are uniquely defined by the formula:

$$dv = dx^k \cdot D_k(v). \quad (5)$$

These maps satisfy the relations

$$D_k(x^i) = \delta_k^i, \quad (6)$$

and twisted derivation property

$$D_k(uv) = D_k(u)v + A(u)_k^i D_i(v). \quad (7)$$

By the Leibniz rule (1) we have

$$v dx^i = d(vx^i) - d(v)x^i = dx^k [D_k(vx^i) - D_k(v)x^i]$$

therefore, the partial derivatives D_k and the homomorphism A are connected by the relations

$$A(v)_k^i = D_k(vx^i) - D_k(v)x^i \quad (8)$$

This shows that for a given coordinate calculus a left module structure on the right free module $V \otimes R$ is uniquely determined by (8). A natural question concerning an inverse problem arises here. If a homomorphism A is given, then the formula (7) allows one to calculate partial derivatives of a product in terms of its factors. That fact and formula (6) show that for a given A there exists not more then one coordinate calculus with partial derivatives satisfying the formulae (6) and (7). It is not clear yet whether or not there exists at least one differential of such a type. Thus, our first task is to describe these homomorphisms A for which there exist coordinate differentials.

Theorem 2.3 *Let $\hat{R} = \mathbb{F} \langle x^1, \dots, x^n \rangle$ be a free algebra generated by x^1, \dots, x^n and A^1, \dots, A^n be any set of $n \times n$ matrices over \hat{R} . There exists the unique coordinate differential d_A of \hat{R} such that $A(x^i) = A^i$ i.e. such that the following commutation rules are satisfied*

$$x^i dx^j = dx^k \cdot (A^i)_k^j.$$

Let now R be a non-free algebra defined by the set of generators x^1, \dots, x^n and an ideal $I \subset \hat{R}$ of relations such that $R = \hat{R}/I$.

Definition 2.4 An ideal $I \neq \hat{R}$ of a free algebra $\hat{R} = \mathbb{F} \langle x^1, \dots, x^n \rangle$ is said to be *consistent* with a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ if the the following two consistency condition are satisfied:

$$A(I)_k^i \subseteq I \tag{C1}$$

$$D_k(I) \subseteq I \tag{C2}$$

where, D_k are partial derivatives for the differential d_A (cf. Theorem 2.3).

These two conditions generalize Wess and Zumino quadratic and linear consistency condition for quadratic algebras.

Theorem 2.5 *Let an ideal $I \neq \hat{R}$ of the free algebra $\hat{R} = \mathbb{F} \langle x^1, \dots, x^n \rangle$ be consistent with a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$. Then the quotient algebra $R = \hat{R}/I$ has a coordinate differential with the homomorphis $\pi(A) : R \rightarrow R_{n \times n}$, where $\pi : \hat{R} \rightarrow R$ is a canonical epimorphism.*

Corollary 2.6 *Let an ideal $I \neq \hat{R}$ of the free algebra $\hat{R} = \mathbb{F} \langle x^1, \dots, x^n \rangle$ be generated by the set of homogeneous relations $\{f_a \in \hat{R}, a \in \mathcal{M}\}$ of the same degree. Assume that the homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ acts linearly on generators i.e. $A(x^i)_k^j = \alpha_{kl}^{ij} x^l$. Then the ideal I is A -consistent or equivalently the quotient algebra $R = \hat{R}/\langle f_a = 0, a \in \mathcal{M} \rangle$ has a coordinate differential with the commutation rule $x^i dx^j = \alpha_{kl}^{ij} dx^k \cdot x^l$ if and only if*

$$A(f_a)_k^i = \gamma_{ka}^{ib} f_b \tag{C1'}$$

$$D_k f_a = 0 \quad (C2')$$

for some scalar coefficients γ_{ka}^{ib} .

We have proved that a free algebra \hat{R} admits a coordinate calculus for arbitrary commutation rules. In order to define a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ it is enough to set its value on generators (cf. Theorem 2.3)

$$(A^m)_k^i \equiv A(x^m)_k^i = \alpha_k^{mi} + \alpha_{kj_1}^{mi} x^{j_1} + \alpha_{kj_1 j_2}^{mi} x^{j_1} x^{j_2} + \dots$$

where, $\{\alpha_{kj_1, \dots, j_r}^{mi} \in \mathbb{F}\}$ are arbitrary tensor coefficients. If the homomorphism A preserves a degree, then it must act linearly on generators i.e., $A(x^m)_k^i = \alpha_{kj}^{mi} x^j$. Therefore, the homomorphism A is defined by the 2-covariant 2-contravariant tensor $A = \alpha_{kj}^{mi}$. This is a homogeneous case, the most frequently studied in the literature. In the case $A(x^m)_k^i = \alpha_k^{mi} + \alpha_{kj}^{mi} x^j$ the partial derivatives do not increase a degree. The case $A(x^m)_k^i = \alpha_k^{mi} + \delta_k^i x^m$ has been considered by Dimakis and Müller–Hoissen [4].

For any homomorphism A there exists the largest A -consistent ideal $I(A)$ contained in the ideal \bar{R} of polynomials with zero constant terms ($\bar{R} \oplus \mathbb{F} \cdot 1 = \hat{R}$) – the sum of all consistent ideals of such a type. Note that the ideal $I(A)$ need not to be the only maximal A -consistent ideal in \bar{R} .

Definition 2.7 The factor algebra $R(A) = \hat{R}/I(A)$ is called an *optimal* algebra for a commutation rule $x^i dx^j = dx^k \cdot A(x^i)_k^j$.

We shall describe the ideal $I(A)$ in the homogeneous case.

Theorem 2.8 For any 2-covariant 2-contravariant tensor $A = \alpha_{kl}^{ij}$ the ideal $I(A)$ can be constructed by induction as the homogeneous space $I(A) = I_1(A) + I_2(A) + I_3(A) + \dots$ in the following way:

1. $I_1(A) = 0$
2. Assume that $I_{s-1}(A)$ has been defined and U_s be a space of all polynomials m of degree s such that $D_k(m) \in I_{s-1}(A)$ for all k . $1 \leq k \leq n$. Then $I_s(A)$ is the largest A -invariant subspace of U_s .

The ideal $I(A)$ is a maximal A -consistent ideal in \hat{R} .

We have proved that there exists a maximal algebra which has a coordinate differential with an arbitrary commutation rule. This is a free algebra \hat{R} . In general, there exist many algebras which have coordinate calculi with a given commutation rule A . Each of them has the form \hat{R}/I where, I is an A -consistent ideal in \hat{R} . The optimal algebra $R(A)$ is a minimal one in the following sense: it does not contain A -consistent ideals at all. In the homogeneous case this algebra is characterized as the

unique algebra which has no nonzero A -invariant subspaces with zero differentials. We will see from the examples below that the same algebra can be an optimal algebra for various commutation rules.

3 Optimal versus quadratic algebras.

In this section we shall address the existence problem for calculi on quadratic algebras. It turns out that this is closely related to quadratic identities in the optimal algebra.

A coordinate algebra on V is *quadratic* if it is the quotient of the tensor algebra TV by an ideal $\langle W \rangle$ generated by a subspace $W \subset V \otimes V$. It is customary and convenient to consider the subspace W in the form

$$W_B = \text{lin}(x^i \otimes x^j - \beta_{kl}^{ij} x^k \otimes x^l) \quad (9)$$

with β_{kl}^{ij} being a matrix of some endomorphism B of $V \otimes V$. Elements of $\text{End}(V \otimes V)$ we shall call *twists*. We will denote by R_B the quadratic algebra associated with the twist B

$$R_B = TV / \langle W_B \rangle \cong \mathbb{F} \langle x^1, \dots, x^n \rangle / \langle x^i x^j = \beta_{kl}^{ij} x^k x^l \rangle. \quad (10)$$

This method of presentation of the algebra R_B has disadvantage of not being unique: a huge number of twists lead to the same algebra. For example, if $\bar{W} \oplus W = V \otimes V$ is the direct sum decomposition then the projection onto W is a good candidate for B (cf. [11]). It is not our purpose to develop this point here. An interesting case appear when the twist B satisfies the Yang–Baxter equation

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23}. \quad (11)$$

In the sequel we assume the homogeneous commutation rule $x^i dx^j = \alpha_{kl}^{ij} dx^k \cdot x^l$ is given. In this case with a homomorphism $A : \hat{R} \rightarrow \hat{R}_{n \times n}$ one can associate a twist determined by the same 2×2 tensor α_{kl}^{ij} . By abuse of notation we shall denote this endomorphism by the same letter, $A \in \text{End}(V \otimes V)$.

This yields the consistency problem between the ideal $\langle W_B \rangle$ and the commutation rule A . The following proposition provides a convenient criterion for the consistency check.

Proposition 3.1 *Let A, B be twists on V . Then $\langle W_B \rangle$ is an A -consistent ideal in TV if and only if the following two consistency conditions are satisfied: the generalized Yang–Baxter equation*

$$A_{12}A_{23}(E - B)_{12} = (E - B)_{23}Z \quad (C1'')$$

has a solution in a 3×3 tensor $Z \in \text{End}(V \otimes V \otimes V)$, and

$$(E - B)(E + A) = 0 \quad (C2'')$$

where, E is the identity twist.

Proof: The proof follows directly from the Corollary 2.6 . \square

Every solution of the consistency conditions (C1'') and (C2'') leads to a coordinate calculus on the algebra R_B with the commutation rule A . In particular, we have $W_B \subseteq I_2(A)$ and $\langle W_B \rangle \subseteq I(A)$ (cf. Theorem 2.8). One can express it by saying that the quadratic relations $x^i x^j = \beta_{kl}^{ij} x^k x^l$ are fulfilled in the optimal algebra $R(A)$. In other words, all solutions of the consistency conditions (C1'') and (C2'') allow us to find out all possible (quadratic) identities in the optimal algebra $R(A)$. But in general, the quadratic algebra R_B is not an optimal algebra for the commutation rule A .

Remark 3.2 If the differential d_A on R_B is *nondegenerate* in the following sense: the kernel of d_A consists only the ground field \mathbb{F} then $R(A) = R_B$.

One can also pose an inverse problem. Assume that the quadratic algebra R_B is given and we are looking for solutions A, Z of the consistency conditions. It allows us to classify all existing homogeneous calculi on the quadratic algebra R_B , or equivalently to classify all homogeneous commutation rules for which optimal algebras satisfy the relations $x^i x^j = \beta_{kl}^{ij} x^k x^l$. For example, in [2] we have classified all homogeneous commutation rules in two variables for which the optimal algebras are commutative.

Set A, Z arbitrary and $B = E$. These are trivial solutions of the consistency conditions (C1''), (C2''). In this case $R_E \equiv TV$. It restates in the homogeneous case our previous result that the free algebra admits a coordinate differential with any commutation rule A .

Wess and Zumino have found another solutions. If twists A, B satisfy the Yang–Baxter equation of the form

$$A_{12}A_{23}B_{12} = B_{23}A_{12}A_{23} \quad (12)$$

then they solve the quadratic consistency condition (C1''). (Indeed, in this case $Z = A_{12}A_{23}$). In particular, if A satisfies the Yang–Baxter equation (11) itself then $B = \mu A$, $\mu \in \mathbb{F}$ also is a solution of (C1''). The linear consistency condition (C2'') now becomes a minimal polynomial condition for A

$$(E/\mu - A)(E + A) = 0. \quad (13)$$

This is the Hecke equation for A .

More general solutions with the higher order minimal polynomial were found by Hlavaty [6]. Let A be a Yang–Baxter twist as previously. Assume $-1 \in \text{Spec} A$ and $G(A)(E + A) = 0$ is some (possibly minimal) polynomial identity satisfied by A , where $G(A) = \sum \mu_k A^k$ is a polynomial in the matrix A . Then $B = E - G(A)$ and $Z = A_{12}A_{23}$ satisfy both consistency conditions.

Example 3.3 (*Manin's spaces*) Consider a twist B in the form

$$B(x^i \otimes x^j) = \beta^{ji} x^j \otimes x^i \quad (14)$$

defined by some n by n matrix β^{ij} . In this case $\beta_{kl}^{ij} = \beta^{ji} \delta_l^i \delta_k^j$. This choice leads to the various Manin's spaces determined by the coefficients β^{ij} (cf. [8]).

Remark 3.4 This is an easy observation that arbitrary three twists A, B, C each of the form (14) do satisfy the Yang–Baxter equation $A_{12}C_{23}B_{12} = B_{23}C_{12}A_{23}$.

Let us consider the commutation rule A of the form (14), i.e.: $x^j dx^i = dx^i \cdot \alpha^{ij} x^j$, with some matrix α^{ij} . Look for quadratic relations given by the formula (14) in the optimal algebra. Due to the Remark 3.4 the quadratic consistency condition (C1'') is automatically satisfied in the form (12). Substituting to the linear condition (C2'') one gets

$$(\alpha^{ji} - \beta^{ji}) \delta_m^j \delta_p^i + (1 - \beta^{ji} \alpha^{ij}) \delta_m^i \delta_p^j = 0$$

Consider two cases. For $i = j$ we have the condition

$$(\alpha^{ii} + 1)(1 - \beta^{ii}) = 0.$$

It means that β^{ii} has to be 1 whenever $\alpha^{ii} \neq -1$.

For $i \neq j$ we obtain two conditions

$$\alpha^{ij} = \beta^{ij} \quad \text{and} \quad \alpha^{ij} \alpha^{ji} = 1.$$

Thus we get the following result. If the matrix α^{ij} does satisfy the property $\alpha^{ij} \alpha^{ji} = 1$ for $i \neq j$ then $\beta^{ij} = \alpha^{ij}$ for $i \neq j$ and $\beta^{ii} = 1$ solves both consistency conditions. In this case the relations $\alpha^{ij} x^i x^j = x^j x^i$ for $i \neq j$ are satisfied in the optimal algebra $R(A)$. It means that the Manin space

$$R_B = \mathbb{F} \langle x^1, \dots, x^n \rangle / \langle \alpha^{ij} x^i x^j = x^j x^i, i < j \rangle \quad (15)$$

have a coordinate calculus with our commutation rule A . For this algebra to be optimal by the Remark 3.2 it is enough to check that this differential is nondegenerate. It was done in [2]. Summarizing, we have

Example 3.5 (*Calculi on Manin's spaces*) Consider the diagonal commutation rule $x^j dx^i = \alpha^{ij} dx^i \cdot x^j$ with $\alpha^{ij} \alpha^{ji} = 1$, for $i \neq j$. There are two cases.

- (A) If none of the coefficients q^{ii} is a root of a polynomial of the type $\lambda^{[m]} \doteq \lambda^{m-1} + \lambda^{m-2} + \dots + 1$ then the optimal algebra $R(A)$ for this commutation rule has the form (15).
- (B) If $(\alpha^{ii})^{[m_i]} = 0$, $1 \leq i \leq s$ with the minimal m_i then $R(A) = R_B / \langle (x^i)^{m_i} = 0, 1 \leq i \leq s \rangle$.

$$\mathbb{F} \langle x^1, \dots, x^n \rangle / \langle \alpha^{ij} x^i x^j = x^j x^i, i < j, (x^i)^{m_i} = 0, 1 \leq i \leq s \rangle \quad (16)$$

It should be noted that in the case (B) the optimal algebra is not in general a quadratic algebra. The paragrassmann case $m_i = m > 2$ has been studied in [5].

Observe that in the case (A), the diagonal elements α^{ii} have no influence on the form (15) of the algebra $R(A)$. But different α^{ii} lead to different commutation rule and different differentials. This is the easiest way to see that the same algebra can be an optimal algebra for different calculi.

It is rather expected result that the polynomial algebra (free commutative algebra) is an optimal algebra for the Newton–Leibniz calculus ($\alpha^{ij} = 1$), while the Grassmann algebra is optimal for the supersymmetric calculus ($\alpha^{ij} = -1$).

4 Higher order calculi.

Definition 4.1. A graded differential algebra (GDA for short) is a \mathbb{N} -graded algebra

$$\Omega = \oplus_{m \in \mathbb{N}} \Omega^m$$

and a homogeneous \mathbb{F} -linear mapping d of degree 1

$$d : \Omega \rightarrow \Omega, \quad d = \oplus_{m \in \mathbb{N}} d_m, \quad d_m : \Omega^m \rightarrow \Omega^{m+1}$$

such that:

$$d^2 = 0$$

$$d(\omega_1 \cdot \omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{\deg \omega_1} \omega_1 \cdot d\omega_2$$

for each $\omega_1, \omega_2 \in \Omega$, ω_1 homogeneous.

In particular, it follows from the above definition that Ω^0 is an algebra, Ω is an Ω^0 -algebra, each Ω^n is an Ω^0 -bimodule and $d_0 : \Omega^0 \rightarrow \Omega^1$ is a (first order) differential. This fact can be expressed by saying that the GDA Ω is an extension of the first order calculus d_0 to higher order. Since GDA is also a complex ($d^2 = 0$) one can think of it as a noncommutative or quantum generalization of a de Rham complex. Conversely, if the first order differential $d : R \rightarrow M$ is given, there exist in general many extension of it to a GDA Ω in such a way that $\Omega^0 = R$, $\Omega^1 = M$ and $d_0 = d$. Several such constructions have been proposed in the literature.

Example 4.2. Wornowicz’s external algebra formalism for quantum groups [13].

Example 4.3. (*Universal GDAs*) Let $d : R \rightarrow M$ be a first order differential such that the bimodule M is generated by differentials i.e. $M = dR \cdot R$. Maltsiniotis [7] has shown that there exists a GDA $\Omega(R, M)$ which satisfies certain universal property: each GDA extension of $d : R \rightarrow M = dR \cdot R$ can be obtained from $\Omega(R, M)$ via a standard quotient construction. In particular, if $d : R \rightarrow M(R)$ is a universal first

order differential (cf. Example 2.2) then the universal GDA $\Omega(R) \equiv \Omega(R, M(R))$ is known as a differential envelope of the algebra R .

Here, we propose a general construction of GDA $\Omega_A(R, M)$ for a coordinate differential $d : R \rightarrow M$ with a commutation rule $x^i dx^j = dx^k \cdot A(x^i)_k^j$. This construction is related to an external algebra formalism. The detailed discussion and proof will appear in a forthcoming publication [3].

Let $T_R M = R \oplus M \oplus M \otimes_R M \oplus \dots$ be a tensor algebra of the bimodule M . $T_R M$ is a free right module freely generated by elements of the form $dx^{i_1} \otimes \dots \otimes dx^{i_m}$, $m \in \mathbb{N}$, i.e. $T_R M = T(dV) \otimes_{\mathbb{F}} R$ where, $T(dV)$ is a tensor algebra of the vector space $dV = \text{lin}(dx^1, \dots, dx^n)$. Define a linear mapping $d : T_R M \rightarrow T_R M$ by

$$d(dx^{i_1} \otimes \dots \otimes dx^{i_m} \cdot v_{i_1 \dots i_m}) = (-1)^m dx^{i_1} \otimes \dots \otimes dx^{i_m} \otimes dv_{i_1 \dots i_m} \quad (17)$$

Note that $(T_R M, d)$ is not a GDA.

Theorem 4.4. *Let J_A denotes an R -homogeneous ideal in the R -algebra $T_R M$ generated by the n^2 elements*

$$dx^i \otimes dx^j + dx^k \otimes dx^l \cdot D_l(A(x^i)_k^j)$$

Let $\Omega_A(R, M) = T_R M / J_A$. Then

1. (GDA)

$$\Omega_R(R, M) = R \oplus M \oplus \Omega_A^2(R, M) \oplus \dots$$

is a GDA where, now d is a mapping induced from (17).

2. (existence of wedge product)

In the homogeneous case $A(x^i)_k^j = \alpha_{kl}^{ij} x^l$ one has

$$\Omega_A^m(R, M) = \Lambda_A^m(dV) \cdot R$$

i.e. $\Omega_A^m(R, M)$ is generated by elements from $\Lambda_A^m(dV)$ where,

$$\Lambda_A(dV) = \oplus_{m \in \mathbb{N}} \Lambda_A^m(dV) = T(dV) / \langle dx^i \otimes dx^j + dx^k \otimes dx^l \alpha_{kl}^{ji} \rangle$$

3. (external algebra structure)

Moreover, the equality

$$\Omega_A^m(R, M) = \Lambda_A^m(dV) \otimes_{\mathbb{F}} R$$

holds, i.e. the bimodule $\Omega_A^m(R, M)$ is a free right module if and only if the following generalized Yang-Baxter equation has a solution in a 3×3 tensor Y :

$$A_{12}A_{23}A_{12} - A_{23}A_{12}A_{23} = (E + A)_{12}Y$$

here, $A = \alpha_{kl}^{ij}$.

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